

# On a C. de Boor's Conjecture in a Particular Case and Related Perturbation

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## Abstract

In this paper, we focus on two classes of  $D$ -invariant polynomial subspaces. The first is a classical type, while the second is a new class. With matrix computation, we prove that every ideal projector with each  $D$ -invariant subspace belonging to either the first class or the second is the pointwise limit of Lagrange projectors. This verifies a particular case of a C. de Boor's conjecture asserting that every complex ideal projector is the pointwise limit of Lagrange projectors. Specifically, we provide the concrete perturbation procedure for ideal projectors of this type.

*Key words:* Ideal projector, Lagrange projector, Pointwise limit, C. de Boor's conjecture

*2000 MSC:* 41A63, 41A10, 41A35

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## 1. Introduction

Polynomial interpolation is to construct a polynomial  $p$  belonging to a finite-dimensional subspace of  $\mathbb{F}[\mathbf{x}]$  from a set of data that agrees with a given function  $f$  at the data set, where  $\mathbb{F}[\mathbf{x}] := \mathbb{F}[x_1, \dots, x_d]$  denotes the polynomial ring in  $d$  variables over the field  $\mathbb{F}$ . It's important to make the comment that  $\mathbb{F}$  is the real field  $\mathbb{R}$  or the complex field  $\mathbb{C}$  in this paper. Univariate polynomial interpolation has a well developed theory, while the multivariate one is very problematic since a multivariate interpolation polynomial is determined not only by the cardinal but also by the geometry of the data set, cf. [1, 2].

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Recently, more and more people are getting interested in ideal interpolation, which is defined by an *ideal projector* on  $\mathbb{F}[\mathbf{x}]$ , namely a linear idempotent operator on  $\mathbb{F}[\mathbf{x}]$  whose kernel is an ideal, cf. [3]. When the kernel of an ideal projector  $P$  is the vanishing ideal of certain finite set  $\Xi$  in  $\mathbb{F}^d$ ,  $P$  is a *Lagrange projector* which provides the Lagrange interpolation on  $\Xi$ . Obviously,  $P$  is finite-rank since its range is a  $\#\Xi$ -dimensional subspace of  $\mathbb{F}[\mathbf{x}]$ . Lagrange projectors are the standard examples of ideal projectors.

It's well-known that an ideal projector can be characterized completely by the range of its dual projector, cf. [4, 5, 6, 7]. Given a finite-rank linear projector  $P$  on  $\mathbb{F}[\mathbf{x}]$ , the kernel of  $P$  is an ideal if and only if the range of its dual projector is of the form

$$\bigoplus_{\xi \in \Xi} \delta_{\xi} \mathcal{Q}_{\xi}(D)$$

with some finite point set  $\Xi \subset \mathbb{F}^d$ ,  $D$ -invariant finite-dimensional polynomial subspace  $\mathcal{Q}_{\xi} \subset \mathbb{F}[\mathbf{x}]$  for each  $\xi \in \Xi$ .  $\delta_{\xi}$  denotes the evaluation functional at the point  $\xi$ , and  $\mathcal{Q}_{\xi}(D)$  will be explained in the next section.

In the univariate case, for an integer  $n$ , there is only one  $D$ -invariant polynomial subspace of degree less than  $n$ , which implies that every univariate ideal projector can be viewed as a limiting case of Lagrange projector, cf. [8]. This prompted C. de Boor to define *Hermite projector* as the pointwise limit of Lagrange projectors and pose the following conjecture in [9]. Indeed, this conjecture had been raised in [10] with certain restriction.

**C. de Boor's conjecture** *A finite-rank linear projector on  $\mathbb{C}[\mathbf{x}]$  is an ideal projector if and only if it is the pointwise limit of Lagrange projectors.*

B. Shekhtman [11] constructed a counterexample to this conjecture for every  $d \geq 3$ . In the same paper, B. Shekhtman also showed that the conjecture is true for bivariate complex projectors with the help of Fogarty Theorem (see [12]). Later, using the fact that any pair of commuting matrices can be approximated by pairs of diagonalizable commuting complex matrices (see [13, 14]), C. de Boor and B. Shekhtman [15] reproved the same result. In addition, by Theorem 8 of [16], [15] also proved that certain low-rank multivariate ideal projectors are the limit of Lagrange projectors. Specifically, B. Shekhtman [17] completely analyzed the bivariate ideal projectors which are onto the space of polynomials of degree less than  $n$  over real or complex field, and verified the conjecture in this particular case.

Since for every  $d \geq 3$ , there exist ideal projectors that are not the pointwise limits of Lagrange projectors, the question that what kind of ideal projectors can be perturbed as Lagrange projectors lies ahead. For this purpose, B. Shekhtman [18] theoretically presented a symbolic algorithm which can determine whether an ideal projector is the limit of Lagrange projectors or not. However, as mentioned by this paper, such a method isn't yet feasible in practice.

In the converse case, one wonders how to generate a sequence of Lagrange projectors practically such that this sequence of Lagrange projectors converges pointwise to a given Hermite projector. More deeply, B. Shekhtman raised the following question in [19].

**Problem** *Let  $P$  be an Hermite projector, and  $P_h$  a sequence of Lagrange projectors such that  $P_h$  converges pointwise to  $P$  as  $h$  tends to zero. Then, what's the relationship between the trajectories of the points in varieties of  $\ker P_h$  and  $P$ ?*

In this paper, we deal with two classes of  $D$ -invariant subspaces. The first one is classical, which has been investigated by many literatures such as [10, 20, 21, 22, 23]. The second is some special type, which is inspired by examples in [10, 9, 19]. We construct a group of interpolation point sets corresponding to each class, and establish the relationship between evaluation functionals induced by the point sets and derivative functionals related to the corresponding  $D$ -invariant subspaces. Based on these results, we generate a sequence of Lagrange projectors which converges pointwise to the ideal projector with related  $D$ -invariant subspaces belonging to either the first or the second class. Equivalently, we constructively verify C. de Boor's conjecture for the ideal projector of this type.

The remainder of this paper is organized as follows. The next section is devoted as a preparation for the paper. Section 3 and 4 discuss the first class of  $D$ -invariant subspaces and the second, respectively. Finally, main theorem of this paper is given in Section 5.

## 2. Preliminaries

In this section, we will firstly introduce some notations and review some basic facts about ideal projector. For more details, we refer the reader to [9, 19].

Throughout the paper, we use  $\mathbb{N}$  to stand for the monoid of nonnegative integers and boldface type for tuples with their entries denoted by the same letter with subscripts, for example,  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ . For arbitrary  $\boldsymbol{\alpha} \in \mathbb{N}^d$ , we define  $\|\boldsymbol{\alpha}\|_1 = \alpha_1 + \dots + \alpha_d$  and  $\boldsymbol{\alpha}! = \alpha_1! \dots \alpha_d!$ .

For arbitrary tuples  $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^d$ , we write  $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$  if  $\boldsymbol{\alpha} - \boldsymbol{\beta}$  has only nonnegative entries. In other words,  $\leq$  is the usual product order on  $\mathbb{N}^d$ . A subset  $\mathfrak{d} \subset \mathbb{N}^d$  is called a *lower set* (alternatively, *down set*, *order ideal*, etc.) if it is *closed* under  $\leq$ , that is,  $\boldsymbol{\alpha} \in \mathfrak{d}$  implies  $\boldsymbol{\beta} \in \mathfrak{d}$  for all  $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$ .

Let  $P$  be a finite-rank ideal projector on  $\mathbb{F}[\boldsymbol{x}]$ . The range and kernel of  $P$  will be described as

$$\begin{aligned} \text{ran}P &:= P(\mathbb{F}[\boldsymbol{x}]) = \{g \in \mathbb{F}[\boldsymbol{x}] : g = Pf \text{ for some } f \in \mathbb{F}[\boldsymbol{x}]\}, \\ \text{ker}P &:= \text{ran}(I - P) = \{g \in \mathbb{F}[\boldsymbol{x}] : Pg = 0\}, \end{aligned}$$

where  $\text{ran}P$  forms a finite-dimensional subspace and  $\text{ker}P$  a zero-dimensional ideal of  $\mathbb{F}[\boldsymbol{x}]$ . It is obvious that the ideal  $\text{ker}P$  complements the subspace  $\text{ran}P$ , i.e.,  $\text{ker}P \oplus \text{ran}P = \mathbb{F}[\boldsymbol{x}]$ .

Furthermore, as an infinite dimensional  $\mathbb{F}$ -vector space,  $\mathbb{F}[\boldsymbol{x}]$  has an algebraic dual  $\mathbb{F}[\boldsymbol{x}]'$ . An ideal projector  $P$  on  $\mathbb{F}[\boldsymbol{x}]$  also has a dual projector  $P'$  on  $\mathbb{F}[\boldsymbol{x}]'$ , and

$$\text{ran}P' = (\text{ker}P)^\perp := \{\lambda \in \mathbb{F}[\boldsymbol{x}]' : \text{ker}P \subset \text{ker}\lambda\}.$$

In fact,  $\text{ran}P'$  is the set of interpolation conditions matched by  $P$ . It is easy to see that the maximum number of linearly independent functionals in  $\text{ran}P'$  equals the dimension of  $\text{ran}P$ , namely  $\dim \text{ran}P = \dim \text{ran}P'$ . In addition, if  $\boldsymbol{q} = (q_1, \dots, q_s)$  is an  $\mathbb{F}$ -basis for  $\text{ran}P$  and  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s)$  an  $\mathbb{F}$ -basis for  $\text{ran}P'$ , then their Gram matrix

$$\boldsymbol{\lambda}^T \boldsymbol{q} := (\lambda_i q_j)_{1 \leq i, j \leq s}$$

is invertible.

For  $\boldsymbol{\alpha} \in \mathbb{N}^d$ , we write  $\boldsymbol{x}^\alpha$  for the monomial  $x_1^{\alpha_1} \dots x_d^{\alpha_d}$ , especially define  $\boldsymbol{x}^\alpha = 1$  for  $\boldsymbol{\alpha} = (0, \dots, 0)$ . Thus, a polynomial  $p$  in  $\mathbb{F}[\boldsymbol{x}]$  can be expressed as

$$p = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^d} \widehat{p}(\boldsymbol{\alpha}) \boldsymbol{x}^\alpha \tag{1}$$

with  $\widehat{p}(\boldsymbol{\alpha}) \in \mathbb{F}$  nonzero. For a polynomial  $p(\mathbf{x})$  given as in (1), we write

$$p(D) : \mathbb{F}[\mathbf{x}] \rightarrow \mathbb{F}[\mathbf{x}]$$

$$f \mapsto \sum_{\boldsymbol{\alpha} \in \mathbb{N}^d} \widehat{p}(\boldsymbol{\alpha}) \frac{\partial^{\|\boldsymbol{\alpha}\|_1} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$$

for the associated differential operator. For a finite-dimensional polynomial subspace  $\mathcal{Q} \subset \mathbb{F}[\mathbf{x}]$ , we define

$$\mathcal{Q}(D) := \text{span}_{\mathbb{F}}\{q(D) : q \in \mathcal{Q}\}.$$

We call a polynomial subspace  $\mathcal{Q}$  a *D-invariant polynomial subspace* if it's closed under differentiation, i.e., for every  $q \in \mathcal{Q}$ ,  $\frac{\partial q}{\partial x_i} \in \mathcal{Q}$  for all  $1 \leq i \leq d$ .

Finally, let us recall some facts about combinatorics. For  $r, m \in \mathbb{N}$  with  $0 \leq r \leq m$ ,  $\binom{m}{r}$  is the binomial coefficient, i.e.,  $\binom{m}{r} = \frac{m!}{r!(m-r)!}$ . For  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ ,  $\binom{\|\boldsymbol{\alpha}\|_1}{\alpha_1, \dots, \alpha_d}$  signifies multinomial coefficient, that is,

$$\binom{\|\boldsymbol{\alpha}\|_1}{\alpha_1, \alpha_2, \dots, \alpha_d} = \frac{\|\boldsymbol{\alpha}\|_1!}{\boldsymbol{\alpha}!}.$$

**Lemma 1.** [24, p. 90] *Let  $n, m$  be arbitrary nonnegative integers satisfying  $n \leq m$ . Then*

$$\sum_{r=0}^m (-1)^{m-r} \binom{m}{r} r^n = \begin{cases} m!, & n = m; \\ 0, & 0 \leq n < m. \end{cases}$$

### 3. The first class of *D*-invariant subspaces

In this section, we are concerned with the classical *D*-invariant subspaces, which only involve directional derivatives. Based on the study of [10, 9, 22], we describe that how the interpolation points are arranged in the straight directions, this sequence of evaluation functionals can converge to the corresponding directional derivative functional.

Let  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$  and  $\boldsymbol{\rho} = (\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \dots, \boldsymbol{\rho}_d) \in (\mathbb{F}^d)^d$  in which

$$\boldsymbol{\rho}_1 = (\rho_{1,1}, \dots, \rho_{1,d}), \boldsymbol{\rho}_2 = (\rho_{2,1}, \dots, \rho_{2,d}), \dots, \boldsymbol{\rho}_d = (\rho_{d,1}, \dots, \rho_{d,d}) \in \mathbb{F}^d$$

are  $\mathbb{F}$ -linearly independent unit vectors. Then we define a differential operator  $D_{\boldsymbol{\rho}}^{\boldsymbol{\alpha}} : \mathbb{F}[\mathbf{x}] \rightarrow \mathbb{F}[\mathbf{x}]$  by the formula

$$D_{\boldsymbol{\rho}}^{\boldsymbol{\alpha}} f = \frac{\partial^{\|\boldsymbol{\alpha}\|_1} f}{\partial \rho_1^{\alpha_1} \cdots \partial \rho_d^{\alpha_d}}.$$

Furthermore, the next lemma tells that for what type of polynomial, its associated differential operator is exactly  $D_{\boldsymbol{\rho}}^{\boldsymbol{\alpha}}$ .

**Lemma 2.** *Let  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ ,  $\boldsymbol{\rho} = (\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \dots, \boldsymbol{\rho}_d) \in (\mathbb{F}^d)^d$  be as above, and let*

$$p = \prod_{i=1}^d (\boldsymbol{\rho}_i \cdot \mathbf{x})^{\alpha_i},$$

*where  $\cdot$  denotes the Euclidean product. Then  $p(D) = D_{\boldsymbol{\rho}}^{\boldsymbol{\alpha}}$ .*

PROOF. The proof can be easily completed by direct computation.  $\square$

**Proposition 3.** *Given a lower set  $\mathfrak{d} \subset \mathbb{N}^d$  and  $\boldsymbol{\rho} = (\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \dots, \boldsymbol{\rho}_d) \in (\mathbb{F}^d)^d$  as above, then the subspace*

$$\mathcal{Q} = \text{span}_{\mathbb{F}} \left\{ \prod_{i=1}^d (\boldsymbol{\rho}_i \cdot \mathbf{x})^{\alpha_i} : \boldsymbol{\alpha} \in \mathfrak{d} \right\} \subset \mathbb{F}[\mathbf{x}].$$

*is  $D$ -invariant.*

PROOF. This proposition is the immediate consequence of the differential calculus.  $\square$

The following proposition makes a connection between difference quotient and directional derivative of multivariate polynomial, which is a variant of Proposition 9.2 of [10] and Proposition 7.2 of [9]. We also provide a simple proof, for completeness.

**Proposition 4.** *Let  $\boldsymbol{\rho} = (\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \dots, \boldsymbol{\rho}_d) \in (\mathbb{F}^d)^d$  be as above,  $h$  a non-zero number in  $\mathbb{F}$ , and let  $\boldsymbol{\alpha} \in \mathbb{N}^d$ ,  $\boldsymbol{\xi} \in \mathbb{F}^d$ . Then for arbitrary  $p \in \mathbb{F}[\mathbf{x}]$ ,*

$$\frac{1}{h^{\|\boldsymbol{\alpha}\|_1}} \sum_{\mathbf{0} \leq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} (-1)^{\|\boldsymbol{\alpha} - \boldsymbol{\beta}\|_1} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} p(\boldsymbol{\xi} + h \sum_{i=1}^d \beta_i \boldsymbol{\rho}_i) = (D_{\boldsymbol{\rho}}^{\boldsymbol{\alpha}} p)(\boldsymbol{\xi}) + O(h), \quad (2)$$

*where  $\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = \prod_{i=1}^d \binom{\alpha_i}{\beta_i}$ , and the remainder  $O(h)$  is a polynomial in  $h$ .*

PROOF. For convenience, let

$$\varphi = \sum_{\mathbf{0} \leq \beta \leq \alpha} (-1)^{\|\alpha - \beta\|_1} \binom{\alpha}{\beta} p(\xi + h \sum_{k=1}^d \beta_k \rho_k).$$

Then we can obtain that for  $m = 0, \dots, \|\alpha\|_1$ ,

$$\begin{aligned} \frac{d^m \varphi(0)}{dh^m} &= \sum_{\|\gamma_1\|_1 + \dots + \|\gamma_d\|_1 = m} \rho_{1,1}^{\gamma_{1,1}} \dots \rho_{d,1}^{\gamma_{d,1}} \dots \rho_{1,d}^{\gamma_{d,1}} \dots \rho_{d,d}^{\gamma_{d,d}} \frac{\partial^m p(\xi)}{\partial x_1^{\|\gamma_1\|_1} \dots \partial x_d^{\|\gamma_d\|_1}} \\ &\quad \left( \begin{matrix} m \\ \gamma_{1,1} \dots \gamma_{1,d}, \dots, \gamma_{d,1}, \dots, \gamma_{d,d} \end{matrix} \right) \prod_{j=1}^d \left( \sum_{\beta_j=0}^{\alpha_j} (-1)^{\alpha_j - \beta_j} \binom{\alpha_j}{\beta_j} \beta_j^{\sum_{i=1}^d \gamma_{i,j}} \right) \end{aligned}$$

with  $\gamma_i = (\gamma_{i,1}, \dots, \gamma_{i,d}) \in \mathbb{N}^d$ ,  $1 \leq i \leq d$ .

In the following, we will simplify the right-hand side of the above equality. There are two cases which must be examined.

Case 1:  $0 \leq m \leq \|\alpha\|_1 - 1$ .

In this case, there must exist some  $1 \leq j \leq d$  such that  $\sum_{i=1}^d \gamma_{i,j} < \alpha_j$ . By Lemma 1, it follows for such  $j$ ,

$$\sum_{\beta_j=0}^{\alpha_j} (-1)^{\alpha_j - \beta_j} \binom{\alpha_j}{\beta_j} \beta_j^{\sum_{i=1}^d \gamma_{i,j}} = 0.$$

Thus, for all  $0 \leq m \leq \|\alpha\|_1 - 1$ ,  $\frac{d^m \varphi(0)}{dh^m} = 0$ .

Case 2:  $m = \|\alpha\|_1$ .

In this case, if there exists some  $1 \leq j \leq d$  such that  $\sum_{i=1}^d \gamma_{i,j} > \alpha_j$ , then there must exist another  $1 \leq j' \leq d$ , such that  $\sum_{i=1}^d \gamma_{i,j'} < \alpha_{j'}$ . Hence, when  $m = \|\alpha\|_1$ ,

$$\prod_{j=1}^d \left( \sum_{\beta_j=0}^{\alpha_j} (-1)^{\alpha_j - \beta_j} \binom{\alpha_j}{\beta_j} \beta_j^{\sum_{i=1}^d \gamma_{i,j}} \right) \neq 0$$

if and only if for all  $1 \leq j \leq d$ ,  $\sum_{i=1}^d \gamma_{i,j} = \alpha_j$ . Combining this with the fact

$$\sum_{\beta_j=0}^{\alpha_j} (-1)^{\alpha_j - \beta_j} \binom{\alpha_j}{\beta_j} \beta_j^{\alpha_j} = \alpha_j!, \quad 1 \leq j \leq d,$$

we can conclude that

$$\begin{aligned} \frac{d^{\|\alpha\|_1} \varphi(0)}{dh^{\|\alpha\|_1}} &= \|\alpha\|_1! \sum_{\gamma_{1,1}+\dots+\gamma_{d,1}=\alpha_1} \dots \sum_{\gamma_{1,d}+\dots+\gamma_{d,d}=\alpha_d} \rho_{1,1}^{\gamma_{1,1}} \dots \rho_{d,1}^{\gamma_{1,d}} \dots \rho_{1,d}^{\gamma_{d,1}} \dots \rho_{d,d}^{\gamma_{d,d}} \\ &\quad \left( \begin{matrix} \alpha_1 \\ \gamma_{1,1}, \dots, \gamma_{d,1} \end{matrix} \right) \dots \left( \begin{matrix} \alpha_d \\ \gamma_{1,d}, \dots, \gamma_{d,d} \end{matrix} \right) \frac{\partial^{\|\alpha\|_1} p(\xi)}{\partial x_1^{\gamma_{1,1}} \dots \partial x_d^{\gamma_{d,1}}}. \end{aligned}$$

That is,

$$\frac{d^{\|\alpha\|_1} \varphi(0)}{dh^{\|\alpha\|_1}} = \|\alpha\|_1! (D_{\rho}^{\alpha} p)(\xi).$$

In sum, we have deduced that

$$\frac{d^m \varphi(0)}{dh^m} = \begin{cases} 0, & 0 \leq m \leq \|\alpha\|_1 - 1; \\ \|\alpha\|_1! (D_{\rho}^{\alpha} p)(\xi), & m = \|\alpha\|_1, \end{cases}$$

which implies that

$$\varphi = h^{\|\alpha\|_1} (D_{\rho}^{\alpha} p)(\xi) + O(h^{\|\alpha\|_1+1}).$$

Equality (2) follows directly from the above equality.  $\square$

#### 4. The second class of $D$ -invariant subspaces

In this section, we present a new class of  $D$ -invariant subspaces, which are spanned by polynomials with special structure. We also provide a group of interpolation point sets corresponding to this class of  $D$ -invariant subspaces, and discuss that how some interpolation points coalesce in the non-straight directions, such sequence of evaluation functionals can converge to the differential functional about this class of  $D$ -invariant subspaces.

Before introducing the new class of  $D$ -invariant subspaces, we need to settle some notations used throughout this section.

Let  $\mathbf{a} = (a_0, a_1, \dots, a_n)$  with  $n \geq 1$  be an  $n+1$ -tuple of positive integers satisfying

$$a_0 = 1 \text{ and } a_1 > \dots > a_n \geq 2, \quad (3)$$

and let

$$\mathbf{c}_1 = (c_{1,0}, c_{1,1}, \dots, c_{1,n}), \dots, \mathbf{c}_d = (c_{d,0}, c_{d,1}, \dots, c_{d,n}) \in \mathbb{F}^{n+1}$$



satisfying that  $c_{1,0}, c_{2,0}, \dots, c_{d,0}$  aren't all zero. Then it's clear that

$$\tau(\gamma_1, \dots, \gamma_d) = \sum_{j=0}^n a_j \sum_{i=1}^d \gamma_{i,j} \quad (4)$$

with  $\gamma_i = (\gamma_{i,0}, \gamma_{i,1}, \dots, \gamma_{i,n}) \in \mathbb{N}^{n+1}$  defines a map  $\tau : (\mathbb{N}^{n+1})^d \rightarrow \mathbb{N}$ .

**Proposition 5.** *Let  $\mathbf{a} = (a_0, a_1, \dots, a_n)$ ,  $\mathbf{c}_i = (c_{i,0}, c_{i,1}, \dots, c_{i,n})$ ,  $1 \leq i \leq d$  and the map  $\tau$  be as above. Let  $q_{n,m}$ ,  $m = 0, 1, \dots, a_1$  be polynomials defined by*

$$q_{n,m} = \sum_{\tau(\gamma_1, \dots, \gamma_d) = m} \frac{\mathbf{c}_1^{\gamma_1} \dots \mathbf{c}_d^{\gamma_d}}{\gamma_1! \dots \gamma_d!} x_1^{\|\gamma_1\|_1} \dots x_d^{\|\gamma_d\|_1},$$

with  $\gamma_i = (\gamma_{i,0}, \gamma_{i,1}, \dots, \gamma_{i,n}) \in \mathbb{N}^{n+1}$ ,  $\mathbf{c}_i^{\gamma_i} = \prod_{j=0}^n c_{i,j}^{\gamma_{i,j}}$ , and let  $\mathcal{Q}$  be a polynomial subspace defined by

$$\mathcal{Q} = \text{span}_{\mathbb{F}}\{q_{n,m} : 0 \leq m \leq a_1\}.$$

Then the following hold:

- (i)  $\dim \mathcal{Q} = a_1 + 1$ ,
- (ii)  $\mathcal{Q}$  is a  $D$ -invariant polynomial subspace.

PROOF. To prove (i), we assume that there exist  $k_0, k_1, \dots, k_{a_1-1}, k_{a_1} \in \mathbb{F}$  such that

$$k_0 q_{n,0} + k_1 q_{n,1} + \dots + k_{a_1-1} q_{n,a_1-1} + k_{a_1} q_{n,a_1} = 0.$$

Since there exists some  $1 \leq i_0 \leq d$  such that  $c_{i_0,0} \neq 0$ , then  $x_{i_0}^m$  must belong to the support of  $q_{n,m}$ . Using this together with the definition of  $\tau$ , we conclude that the degree of  $q_{n,m}$  is  $m$ .

Specifically,  $x_{i_0}^{a_1}$  belongs to the support of  $q_{n,a_1}$ , while for all  $0 \leq m \leq a_1 - 1$ ,  $x_{i_0}^{a_1}$  can't belong to the support of  $q_{n,m}$ . Therefore  $k_{a_1} = 0$ , which implies that

$$k_0 q_{n,0} + k_1 q_{n,1} + \dots + k_{a_1-1} q_{n,a_1-1} = 0.$$

Arguing for  $m = a_1 - 1, a_1 - 2, \dots, 1, 0$  as for  $m = a_1$ , we get  $k_{a_1-1} = 0, k_{a_1-2} = 0, \dots, k_1 = 0, k_0 = 0$ , successively. That is to say, the family of polynomials  $q_{n,m}$ ,  $0 \leq m \leq a_1$  is  $\mathbb{F}$ -linearly independent. So  $\dim \mathcal{Q} = a_1 + 1$ .

To prove (ii), we firstly notice that if  $1 \leq m < a_n$ , then  $\tau(\gamma_1, \dots, \gamma_d) = m$  implies that  $\gamma_{i,j} = 0$  with  $1 \leq i \leq d, 1 \leq j \leq n$ . Thus, for all  $1 \leq m < a_n$ ,

$$\begin{aligned} q_{n,m} &= \sum_{\gamma_{1,0} + \dots + \gamma_{d,0} = m} \frac{c_{1,0}^{\gamma_{1,0}} \dots c_{d,0}^{\gamma_{d,0}}}{\gamma_{1,0}! \dots \gamma_{d,0}!} x_1^{\gamma_{1,0}} \dots x_d^{\gamma_{d,0}} \\ &= \frac{1}{m!} (c_{1,0}x_1 + c_{2,0}x_2 + \dots + c_{d,0}x_d)^m. \end{aligned}$$

Consequently,

$$\frac{\partial q_{n,m}}{\partial x_i} = c_{i,0}q_{n,m-1}, \quad 1 \leq m < a_n. \quad (5)$$

Next, let  $m$  be an arbitrary integer satisfying  $a_n \leq m \leq a_1$ , and  $s$  the minimum integer between 1 and  $n$  such that  $m - a_s \geq 0$ . We claim

$$\frac{\partial q_{n,m}}{\partial x_i} = c_{i,0}q_{n,m-1} + \sum_{j=s}^n c_{i,j}q_{n,m-a_j}, \quad a_n \leq m \leq a_1. \quad (6)$$

This claim together with equality (5) immediately means that  $\mathcal{Q}$  is  $D$ -invariant.

To prove our claim, we will use induction on the number  $n$ . When  $n = 1$ , our claim can be easily verified. Now, assume that our claim is true for  $n - 1$ . To prove that it holds for  $n$ , let  $k$  be the maximum nonnegative integer such that  $ka_n \leq m$ , that is,

$$ka_n \leq m, (k+1)a_n > m. \quad (7)$$

and  $k'$  the maximum nonnegative integer such that  $k'a_n \leq m - 1$ . From the definition of  $q_{n,m}$ , we obtain that

$$\begin{aligned} q_{n,m} &= \sum_{l=0}^k \left( \sum_{\tau((\gamma_{1,0}, \dots, \gamma_{1,n-1}), \dots, (\gamma_{d,0}, \dots, \gamma_{d,n-1})) = m - la_n} \frac{c_{1,0}^{\gamma_{1,0}} \dots c_{1,n-1}^{\gamma_{1,n-1}} \dots c_{d,0}^{\gamma_{d,0}} \dots c_{d,n-1}^{\gamma_{d,n-1}}}{\gamma_{1,0}! \dots \gamma_{1,n-1}! \dots \gamma_{d,0}! \dots \gamma_{d,n-1}!} \right. \\ &\quad \left. x_1^{\gamma_{1,0} + \dots + \gamma_{1,n-1}} \dots x_d^{\gamma_{d,0} + \dots + \gamma_{d,n-1}} \right) \left( \sum_{\gamma_{1,n} + \dots + \gamma_{d,n} = l} \frac{c_{1,n}^{\gamma_{1,n}} \dots c_{d,n}^{\gamma_{d,n}}}{c_{1,n}! \dots c_{d,n}!} x_1^{\gamma_{1,n}} \dots x_d^{\gamma_{d,n}} \right). \end{aligned}$$

That is,

$$q_{n,m} = \sum_{l=0}^k \frac{1}{l!} (c_{1,n}x_1 + \dots + c_{d,n}x_d)^l q_{n-1,m-la_n}, \quad (8)$$

which plays an important role in what follows. At this point, we have two cases to consider.

Case 1:  $a_n \leq m < a_{n-1}$ .

In this case,  $s = n$  and  $0 \leq m - la_n < a_{n-1}$  for all  $0 \leq l \leq k$ . Hence,

$$q_{n-1, m-la_n} = \frac{1}{(m-la_n)!} (c_{1,0}x_1 + \dots + c_{d,0}x_d)^{m-la_n}, \quad 0 \leq l \leq k.$$

By (8), we get

$$q_{n,m} = \sum_{l=0}^k \frac{1}{l!(m-la_n)!} (c_{1,n}x_1 + \dots + c_{d,n}x_d)^l (c_{1,0}x_1 + \dots + c_{d,0}x_d)^{m-la_n}.$$

So we obtain that

$$\begin{aligned} & \frac{\partial q_{n,m}}{\partial x_i} \\ &= c_{i,0} \sum_{l=0}^{k'} \frac{1}{l!(m-1-la_n)!} (c_{1,n}x_1 + \dots + c_{d,n}x_d)^l (c_{1,0}x_1 + \dots + c_{d,0}x_d)^{m-1-la_n} + \\ & \quad c_{i,n} \sum_{l=1}^k \frac{1}{(l-1)!(m-la_n)!} (c_{1,n}x_1 + \dots + c_{d,n}x_d)^{l-1} (c_{1,0}x_1 + \dots + c_{d,0}x_d)^{m-la_n} \\ &= c_{i,0}q_{n,m-1} + c_{i,n}q_{n,m-a_n}. \end{aligned}$$

Case 2:  $a_{n-1} \leq m \leq a_1$ .

In this case,  $0 \leq s \leq n-1$ .

By (7) and (3), it follows that  $m - ka_n - a_{n-1} < 0$ . Thus, there exists some  $0 \leq l_0 \leq k-1$  such that

$$m - l_0a_n - a_{n-1} \geq 0, m - (l_0 + 1)a_n - a_{n-1} < 0. \quad (9)$$

As a result, we have that  $0 \leq m - la_n < a_{n-1}$  for all  $l_0 + 1 \leq l \leq k$ , which implies that

$$q_{n-1, m-la_n} = \frac{1}{(m-la_n)!} (c_{1,0}x_1 + \dots + c_{d,0}x_d)^{m-la_n}, \quad l_0 + 1 \leq l \leq k.$$

Therefore,

$$\frac{\partial q_{n-1, m-la_n}}{\partial x_i} = \begin{cases} c_{i,0}q_{n-1, m-1-la_n}, & l_0 + 1 \leq l \leq k'; \\ 0, & k' + 1 \leq l \leq k. \end{cases} \quad (10)$$

For all  $0 \leq l \leq l_0$ , we have  $a_{n-1} \leq m - la_n \leq a_1$ . Then our inductive hypothesis implies that

$$\frac{\partial q_{n-1, m-la_n}}{\partial x_i} = c_{i,0} q_{n-1, m-1-la_n} + \sum_{j=s_l}^{n-1} c_{i,j} q_{n-1, m-la_n-a_j}, \quad 0 \leq l \leq l_0, \quad (11)$$

where  $s_l$  is the minimal integer between 1 and  $n-1$  such that  $m-la_n-a_{s_l} \geq 0$ . It should be noticed that  $s_0 = s$ .

By (10) and (11), we can deduce that

$$\begin{aligned} & \frac{\partial q_{n,m}}{\partial x_i} \\ &= c_{i,0} \left( \sum_{l=0}^{k'} \frac{1}{l!} (c_{1,n}x_1 + \dots + c_{d,n}x_d)^l q_{n-1, m-1-la_n} \right) + \\ & \quad c_{i,n} \left( \sum_{l=1}^k \frac{1}{(l-1)!} (c_{1,n}x_1 + \dots + c_{d,n}x_d)^{l-1} q_{n-1, m-la_n} \right) + \\ & \quad \sum_{l=0}^{l_0} \frac{1}{l!} (c_{1,n}x_2 + \dots + c_{d,n}x_d)^l \sum_{j=s_l}^{n-1} c_{i,j} q_{n-1, m-la_n-a_j} \\ &= c_{i,0} q_{n, m-1} + c_{i,n} q_{n, m-a_n} + \sum_{l=0}^{l_0} \frac{1}{l!} (c_{1,n}x_2 + \dots + c_{d,n}x_d)^l \sum_{j=s_l}^{n-1} c_{i,j} q_{n-1, m-la_n-a_j}. \end{aligned}$$

It remains to show the last row of the above equality and the right-side hand of (6) are equal. More precisely, for each  $s_0 \leq j \leq n-1$ , let  $k_j$  denote the maximum nonnegative integer satisfying  $k_j a_n \leq m - a_j$ , that is,

$$k_j a_n \leq m - a_j, (k_j + 1) a_n > m - a_j. \quad (12)$$

From (9), we know  $k_{n-1} = l_0$ .

Due to (8), we observe that

$$q_{n, m-a_j} = \sum_{l=0}^{k_j} \frac{1}{l!} (c_{2,n}x_2 + \dots + c_{d,n}x_d)^l q_{n-1, m-a_j-la_n}, \quad s_0 \leq j \leq n-1.$$

Furthermore,

$$q_{n, m-a_j} = \sum_{l=0}^{l_0} \frac{1}{l!} (c_{1,n}x_2 + \dots + c_{d,n}x_d)^l q_{n-1, m-a_j-la_n}, \quad s_0 \leq j \leq n-1. \quad (13)$$

Here, we have used the fact that  $q_{n-1,m-a_j-la_n} = 0$  for all  $s_0 \leq j \leq n-1$ ,  $k_j + 1 \leq l \leq l_0$ .

Now, recall that  $s_l, 0 \leq l \leq l_0$  is the minimal integer between 1 and  $n-1$  such that  $m - la_n - a_{s_l} \geq 0$ , then

$$\sum_{j=s_0}^{n-1} c_{i,j} q_{n-1,m-a_j-la_n} = \sum_{j=s_l}^{n-1} c_{i,j} q_{n-1,m-a_j-la_n}, \quad 0 \leq l \leq l_0. \quad (14)$$

According to (13), (14) and the fact  $s = s_0$ , we find that

$$\begin{aligned} \sum_{j=s}^{n-1} c_{i,j} q_{n,m-a_j} &= \sum_{j=s_0}^{n-1} c_{i,j} \sum_{l=0}^{l_0} \frac{1}{l!} (c_{1,n}x_2 + \dots + c_{d,n}x_d)^l q_{n-1,m-a_j-la_n} \\ &= \sum_{l=0}^{l_0} \frac{1}{l!} (c_{1,n}x_2 + \dots + c_{d,n}x_d)^l \left( \sum_{j=s_l}^{n-1} c_{i,j} q_{n-1,m-a_j-la_n} \right). \end{aligned}$$

Consequently,

$$\frac{\partial q_{n,m}}{\partial x_i} = c_{i,0} q_{n,m-1} + \sum_{j=s}^n c_{i,j} q_{n,m-a_j}, \quad a_{n-1} \leq m \leq a_1.$$

□

Next, we will give some simple examples of  $D$ -invariant subspaces as in Proposition 5.

**Example 1.** Let  $c_{i,j} = 0, 1 \leq i \leq d, 1 \leq j \leq n$ , then

$$\mathcal{Q} = \text{span}_{\mathbb{F}} \left\{ \frac{1}{m} (c_{1,0}x_1 + \dots + c_{d,0}x_d)^m : 0 \leq m \leq a_1 \right\},$$

which is spanned by homogenous polynomials. We call this case the trivial one. □

**Example 2.** Let  $n = 1$  and  $c_{1,0} = 1, c_{i,0} = 0$  with  $2 \leq i \leq d$ , then

$$\mathcal{Q} = \text{span}_{\mathbb{F}} \left\{ 1, x_1, \frac{1}{2}x_1^2, \dots, \frac{1}{a_1}x_1^{a_1} + c_{1,1}x_1 + c_{2,1}x_2 + \dots + c_{d,1}x_d \right\}.$$

In this example, we should notice that when  $a_1 = 2, d = 2$  and  $c_{1,1} = 0, c_{2,1} = 1$ , then  $\mathcal{Q} = \text{span}_{\mathbb{F}} \{1, x_1, \frac{1}{2}x_1^2 + x_2\}$ . This is the example that had been discussed by some well-known papers, for instance, [10, p. 301], [9, p. 81], and Illustration 6.1.9 of [19]. □

**Example 3.** Let  $n = 1, \mathbf{a} = (1, 2), d = 3, \mathbf{c}_1 = (1, 0), \mathbf{c}_2 = (1, 1), \mathbf{c}_3 = (0, 1)$ , then

$$\mathcal{Q} = \text{span}_{\mathbb{F}}\{1, x_1 + x_2, \frac{1}{2}x_1^2 + x_1x_2 + \frac{1}{2}x_2^2 + x_2 + x_3\},$$

which will be considered in the next section.  $\square$

The next proposition not only gives us a type of interpolation point sets corresponding to the  $D$ -invariant subspaces as in Proposition 5, but also reveals their relationship.

**Proposition 6.** Let  $\mathbf{a} = (a_0, a_1, \dots, a_n), \mathbf{c}_i = (c_{i,0}, c_{i,1}, \dots, c_{i,n})$  with  $1 \leq i \leq d$ , and let  $q_{n,m}$  with  $0 \leq m \leq a_1$  be as above,  $h$  a non-zero number in  $\mathbb{F}$ . Then for arbitrary  $p \in \mathbb{F}[\mathbf{x}]$  and  $\boldsymbol{\xi} \in \mathbb{F}^d$ ,

$$\begin{aligned} \frac{1}{m!h^m} \sum_{0 \leq r \leq m} (-1)^{m-r} \binom{m}{r} p(\boldsymbol{\xi} + (\sum_{j=0}^n c_{1,j}(rh)^{a_j}, \dots, \sum_{j=0}^n c_{d,j}(rh)^{a_j})) = \\ (q_{n,m}(D)p)(\boldsymbol{\xi}) + O(h). \end{aligned} \quad (15)$$

PROOF. Applying Taylor Formulas, we obtain

$$\begin{aligned} p(\boldsymbol{\xi} + (\sum_{j=0}^n c_{1,j}(rh)^{a_j}, \dots, \sum_{j=0}^n c_{d,j}(rh)^{a_j})) = \\ \sum_{k=0}^{\infty} \sum_{\|\boldsymbol{\gamma}_1\|_1 + \dots + \|\boldsymbol{\gamma}_d\|_1 = k} (rh)^{\sum_{1 \leq i \leq d, 0 \leq j \leq n} a_j \gamma_{i,j}} \frac{\mathbf{c}_1^{\boldsymbol{\gamma}_1} \dots \mathbf{c}_d^{\boldsymbol{\gamma}_d}}{\boldsymbol{\gamma}_1! \dots \boldsymbol{\gamma}_d!} \frac{\partial^k p(\boldsymbol{\xi})}{\partial x_1^{\|\boldsymbol{\gamma}_1\|_1} \dots \partial x_d^{\|\boldsymbol{\gamma}_d\|_1}}, \end{aligned}$$

where  $\mathbf{c}_i^{\boldsymbol{\gamma}_i} = \prod_{j=0}^n c_{i,j}^{\gamma_{i,j}}, \boldsymbol{\gamma}_i = (\gamma_{i,0}, \dots, \gamma_{i,n}) \in \mathbb{N}^{n+1}, 1 \leq i \leq d$ .

By means of the map  $\tau$  defined as in (4), the above equality can be rewritten as

$$\begin{aligned} p(\boldsymbol{\xi} + (\sum_{j=0}^n c_{1,j}(rh)^{a_j}, \dots, \sum_{j=0}^n c_{d,j}(rh)^{a_j})) = \\ \sum_{l=0}^m \sum_{\tau(\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d) = l} (rh)^l \frac{\mathbf{c}_1^{\boldsymbol{\gamma}_1} \dots \mathbf{c}_d^{\boldsymbol{\gamma}_d}}{\boldsymbol{\gamma}_1! \dots \boldsymbol{\gamma}_d!} \frac{\partial^{\|\boldsymbol{\gamma}_1\|_1 + \dots + \|\boldsymbol{\gamma}_d\|_1} p(\boldsymbol{\xi})}{\partial x_1^{\|\boldsymbol{\gamma}_1\|_1} \dots \partial x_d^{\|\boldsymbol{\gamma}_d\|_1}} + O(h^{m+1}), \end{aligned}$$

where  $O(h^{m+1})$  is a polynomial in  $h$ .

Finally, by Lemma 1, we can conclude that

$$\sum_{0 \leq r \leq m} (-1)^{m-r} \binom{m}{r} p(\xi + (\sum_{j=0}^n c_{1,j}(rh)^{a_j}, \dots, \sum_{j=0}^n c_{d,j}(rh)^{a_j})) =$$

$$m!h^m \sum_{\tau(\gamma_1, \dots, \gamma_d)=m} \frac{c_1^{\gamma_1} \dots c_d^{\gamma_d}}{\gamma_1! \dots \gamma_d!} \frac{\partial^{\|\gamma_1\|_1 + \dots + \|\gamma_d\|_1} p(\xi)}{\partial x_1^{\|\gamma_1\|_1} \dots \partial x_d^{\|\gamma_d\|_1}} + O(h^{m+1}),$$

which leads to the proposition immediately.  $\square$

## 5. Main theorem

In this section, we consider a particular type of ideal projectors associated with the above two classes of  $D$ -invariant subspaces, and constructively prove that C. de Boor's conjecture is true for ideal projectors of this type.

First of all, we also introduce the notation that will be adopted in main theorem.

Let  $\xi^{(1)}, \dots, \xi^{(\mu)}, \xi^{(\mu+1)}, \dots, \xi^{(\mu+\nu)} \in \mathbb{F}^d$  be distinct points. For each  $1 \leq k \leq \mu$ , let  $\mathfrak{d}^{(k)} \subset \mathbb{N}^d$  be a lower set and  $\rho^{(k)} = (\rho_1^{(k)}, \dots, \rho_d^{(k)}) \in (\mathbb{F}^d)^d$  be as in Section 3. Likewise, for each  $1 \leq l \leq \nu$ , let  $\mathbf{a}^{(l)} = (a_0^{(l)}, a_1^{(l)}, \dots, a_{n(l)}^{(l)})$ ,  $\mathbf{c}_i^{(l)} = (c_{i,0}^{(l)}, c_{i,1}^{(l)}, \dots, c_{i,n(l)}^{(l)})$ ,  $1 \leq i \leq d$ , and  $q_{n(l),m}^{(l)}$ ,  $0 \leq m \leq a_1^{(l)}$  be as in Section 4.

**Theorem 7.** *With the notation above, let  $P$  be an ideal projector with*

$$\text{ran } P' = \text{span}_{\mathbb{F}} \{ \delta_{\xi^{(k)}} D_{\rho^{(k)}}^{\alpha}, \delta_{\xi^{(\mu+l)}} q_{n(l),m}^{(l)}(D) : \\ \alpha \in \mathfrak{d}^{(k)}, 1 \leq k \leq \mu, 0 \leq m \leq a_1^{(l)}, 1 \leq l \leq \nu \},$$

and let  $P_h$  be a Lagrange projector with

$$\text{ran } P'_h = \text{span}_{\mathbb{F}} \{ \delta_{\xi^{(k)} + h \sum_{i=1}^d \alpha_i \rho_i^{(k)}}, \delta_{\xi^{(\mu+l)} + \phi^{(l)}(mh)} : \\ \alpha \in \mathfrak{d}^{(k)}, 1 \leq k \leq \mu, 0 \leq m \leq a_1^{(l)}, 1 \leq l \leq \nu \},$$

where  $h \in \mathbb{F} \setminus \{0\}$  and

$$\phi^{(l)}(mh) = (\sum_{j=0}^{n(l)} c_{1,j}^{(l)}(mh)^{a_j^{(l)}}, \sum_{j=0}^{n(l)} c_{2,j}^{(l)}(mh)^{a_j^{(l)}}, \dots, \sum_{j=0}^{n(l)} c_{d,j}^{(l)}(mh)^{a_j^{(l)}}).$$

Then the following statements hold:

(i) *There exists a positive  $\eta \in \mathbb{F}$  such that*

$$\text{ran} P_h = \text{ran} P, \quad \forall 0 < |h| < \eta.$$

(ii)  *$P$  is the pointwise limit of  $P_h$ ,  $0 < |h| < \eta$ , as  $h$  tends to zero.*

PROOF. Firstly, one can easily verify that

$$\begin{aligned} \boldsymbol{\lambda} &= (\delta_{\boldsymbol{\xi}^{(k)}} D_{\boldsymbol{\rho}^{(k)}}^{\boldsymbol{\alpha}}, \delta_{\boldsymbol{\xi}^{(\mu+l)}} q_{n(l),m}^{(l)}(D)) : \\ \boldsymbol{\alpha} &\in \mathfrak{d}^{(k)}, 1 \leq k \leq \mu, 0 \leq m \leq a_1^{(l)}, 1 \leq l \leq \nu) \in (\mathbb{F}[\boldsymbol{x}'])^s \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\lambda}_h &= (\delta_{\boldsymbol{\xi}^{(k)} + h \sum_{i=1}^d \alpha_i \boldsymbol{\rho}_i^{(k)}}, \delta_{\boldsymbol{\xi}^{(\mu+l)} + \phi^{(l)}(mh)}) : \\ \boldsymbol{\alpha} &\in \mathfrak{d}^{(k)}, 1 \leq k \leq \mu, 0 \leq m \leq a_1^{(l)}, 1 \leq l \leq \nu) \in (\mathbb{F}[\boldsymbol{x}'])^s \end{aligned}$$

form  $\mathbb{F}$ -bases for  $\text{ran} P'$  and  $\text{ran} P'_h$  respectively, where

$$s = \sum_{k=1}^{\mu} \# \mathfrak{d}^{(k)} + \sum_{k=1}^{\nu} a_1^{(k)} + \nu.$$

Provided that the entries of  $\boldsymbol{\lambda}$  and  $\boldsymbol{\lambda}_h$  are arranged in the same order, namely for arbitrary fixed  $1 \leq k \leq \mu$ ,  $\boldsymbol{\alpha} \in \mathfrak{d}^{(k)}$ , the corresponding entries of  $\boldsymbol{\lambda}$  and  $\boldsymbol{\lambda}_h$  are in the same position, the same as for arbitrary fixed  $1 \leq l \leq \nu$ ,  $0 \leq m \leq a_1^{(l)}$ . We denote  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s)$  and  $\boldsymbol{\lambda}_h = (\lambda_{h,1}, \dots, \lambda_{h,s})$ , respectively.

Let  $\boldsymbol{q} = (q_1, q_2, \dots, q_s)$  be an  $\mathbb{F}$ -basis for  $\text{ran} P$ . For convenience, we introduce two  $s \times s$  matrices

$$\boldsymbol{\lambda}^T \boldsymbol{q} := (\lambda_i q_j)_{1 \leq i, j \leq s}, \quad \boldsymbol{\lambda}_h^T \boldsymbol{q} := (\lambda_{h,i} q_j)_{1 \leq i, j \leq s}.$$

and for arbitrary  $f \in \mathbb{F}[\boldsymbol{x}]$ , vectors

$$\boldsymbol{\lambda}^T f := (\lambda_i f)_{1 \leq i \leq s}, \quad \boldsymbol{\lambda}_h^T f := (\lambda_{h,i} f)_{1 \leq i \leq s}.$$

For arbitrary  $p \in \mathbb{F}[\boldsymbol{x}]$ , we have the following facts according to equality (2) and (15).

1. For fixed  $1 \leq k \leq \mu$  and  $\boldsymbol{\alpha} \in \mathfrak{d}^{(k)}$ ,  $(D_{\boldsymbol{\rho}^{(k)}}^{\boldsymbol{\alpha}} p)(\boldsymbol{\xi}^{(k)})$  can be linearly expressed by

$$\{p(\boldsymbol{\xi}^{(k)} + h \sum_{i=1}^d \beta_i \boldsymbol{\rho}_i^{(k)}) : \boldsymbol{\beta} \in \mathfrak{d}^{(k)}\} \cup \{O(h)\}$$

since  $\mathfrak{d}^{(k)}$  is lower, and moreover, the linear combination coefficient of each  $p(\boldsymbol{\xi}^{(k)} + h \sum_{i=1}^d \beta_i \boldsymbol{\rho}_i^{(k)})$  is independent of  $p \in \mathbb{F}[\boldsymbol{x}]$ .



2. For fixed  $1 \leq l \leq \nu$  and  $0 \leq m \leq a_1^{(l)}$ ,  $(q_{n(l),m}^{(l)}(D)p)(\xi^{(\mu+l)})$  can be linearly expressed by

$$\{p(\xi^{(\mu+l)} + \phi^{(l)}(rh)) : 0 \leq r \leq m\} \cup \{O(h)\}.$$

Also, the linear combination coefficient of each  $p(\xi^{(\mu+l)} + \phi^{(l)}(rh))$  is independent of  $p \in \mathbb{F}[\mathbf{x}]$ .

In brief, we can conclude that there exists a nonsingular matrix  $T$  such that

$$\left[ \widehat{\lambda_h^T \mathbf{q}} | \widehat{\lambda_h^T f} \right] := T \left[ \lambda_h^T \mathbf{q} | \lambda_h^T f \right] = \left[ \lambda^T \mathbf{q} | \lambda^T f \right] + [E_h | \epsilon_h], \quad (16)$$

where each entry of  $[E_h | \epsilon_h]$  has the same order as  $h$ . As a consequence, the linear systems

$$\left( \widehat{\lambda_h^T \mathbf{q}} \right) \mathbf{x} = \widehat{\lambda_h^T f} \quad \text{and} \quad (\lambda_h^T \mathbf{q}) \mathbf{x} = \lambda_h^T f$$

are equivalent, namely they have the same set of solutions.

(i) From (16), it follows that each entry of matrix  $\widehat{\lambda_h^T \mathbf{q}}$  converges to its corresponding entry of matrix  $\lambda^T \mathbf{q}$  as  $h$  tends to zero, which implies that

$$\lim_{h \rightarrow 0} \det \left( \widehat{\lambda_h^T \mathbf{q}} \right) = \det (\lambda^T \mathbf{q}).$$

Since  $\det(\lambda^T \mathbf{q}) \neq 0$ , there exists  $\eta > 0$  such that

$$\det \left( \widehat{\lambda_h^T \mathbf{q}} \right) \neq 0, \quad 0 < |h| < \eta.$$

Notice that (16) directly leads to  $\text{rank} \left( \widehat{\lambda_h^T \mathbf{q}} \right) = \text{rank} (\lambda_h^T \mathbf{q})$ ,

$$\text{ran} P_h = \text{span}_{\mathbb{F}} \mathbf{q}, \quad 0 < |h| < \eta,$$

follows, i.e.,  $\mathbf{q}$  forms an  $\mathbb{F}$ -basis for  $\text{ran} P_h$ . Since  $\mathbf{q}$  is also an  $\mathbb{F}$ -basis for  $\text{ran} P$ , we have

$$\text{ran} P = \text{ran} P_h, \quad 0 < |h| < \eta.$$

(ii) Suppose that  $\tilde{\mathbf{x}}$  and  $\mathbf{x}_0$  be the unique solutions of nonsingular linear systems

$$(\lambda_h^T \mathbf{q}) \mathbf{x} = \lambda_h^T f \quad (17)$$

and

$$(\lambda^T \mathbf{q}) \mathbf{x} = \lambda^T f \quad (18)$$

respectively, where  $f \in \mathbb{F}[\mathbf{x}]$  and  $0 < |h| < \eta$ . It is easy to see that

$$P_h f = \mathbf{q} \tilde{\mathbf{x}} \quad \text{and} \quad P f = \mathbf{q} \mathbf{x}_0.$$

Notice that, as  $h \rightarrow 0$ ,  $P$  is the pointwise limit of  $P_h$  if and only if  $P f$  is the coefficientwise limit of  $P_h f$  for all  $f \in \mathbb{F}[\mathbf{x}]$ . Therefore, it is sufficient to show that for every  $f \in \mathbb{F}[\mathbf{x}]$ , the solution vector of system (17) converges to the one of system (18) as  $h$  tends to zero, namely

$$\lim_{h \rightarrow 0} \tilde{\mathbf{x}} = \mathbf{x}_0.$$

By (16), the linear system

$$\left( \widehat{\lambda_h^T \mathbf{q}} \right) \mathbf{x} = \widehat{\lambda_h^T f} \tag{19}$$

can be rewritten as

$$(\lambda^T \mathbf{q} + E_h) \mathbf{x} = (\lambda^T f + \epsilon_h).$$

Since system (19) is equivalent to system (17),  $\tilde{\mathbf{x}}$  is also the unique solution of it. Consequently, using the perturbation analysis of the sensitivity of linear systems (see for example [25, p. 80]), we have

$$\|\tilde{\mathbf{x}} - \mathbf{x}_0\| \leq \left\| (\lambda^T \mathbf{q})^{-1} \right\| \|\epsilon_h - E_h \mathbf{x}_0\| + O(h^2).$$

Since each component of vector  $\epsilon_h - E_h \mathbf{x}_0$  has the same order as  $h$ , it follows that  $\lim_{h \rightarrow 0} \|\tilde{\mathbf{x}} - \mathbf{x}_0\| = 0$ , or, equivalently,  $\lim_{h \rightarrow 0} \tilde{\mathbf{x}} = \mathbf{x}_0$ , which completes the proof of the theorem.  $\square$

Finally, we will give a complete example to illustrate the conclusions of Theorem 7.

**Example 4.** Let  $\boldsymbol{\xi}^{(1)} = (1, 1, 1)$ ,  $\boldsymbol{\xi}^{(2)} = (0, 0, 0)$ . Let

$$\mathbf{q} = (1, x_3, x_2, x_1, x_3^2, x_3 x_2, x_3 x_1)$$

and

$$\begin{aligned} \boldsymbol{\lambda} = & \left( \delta_{\boldsymbol{\xi}^{(1)}}, \delta_{\boldsymbol{\xi}^{(1)}} \frac{\partial}{\partial x_1}, \delta_{\boldsymbol{\xi}^{(1)}} \frac{\partial}{\partial x_2}, \delta_{\boldsymbol{\xi}^{(1)}} \frac{\partial}{\partial x_3}, \delta_{\boldsymbol{\xi}^{(2)}}, \delta_{\boldsymbol{\xi}^{(2)}} \left( \frac{\partial}{\partial x_1} + \frac{\partial f}{\partial x_2} \right), \right. \\ & \left. \delta_{\boldsymbol{\xi}^{(2)}} \left( \frac{1}{2} \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{1}{2} \frac{\partial^2}{\partial x_2^2} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \right) \right) \end{aligned}$$

be the  $\mathbb{F}$ -basis for  $\text{ran}P$  and  $\text{ran}P'$ , respectively.

From Proposition 3 and Example 3, we know that this example is the case of Theorem 7, Therefore, we set

$$\boldsymbol{\lambda}_h = (\delta_{(1,1,1)}, \delta_{(1+h,1,1)}, \delta_{(1,1+h,1)}, \delta_{(1,1,1+h)}, \delta_{(0,0,0)}, \delta_{(h,h^2+h,h^2)}, \delta_{(2h,4h^2+2h,4h^2)}).$$

Recalling the proof of Theorem 7, we can obtain

$$\widehat{\boldsymbol{\lambda}_h^T \mathbf{q}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2+h & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & 1+h & 1 & h^3 & h^2(1+h) & h^2 \\ 0 & 1 & 1 & 0 & 7h^2 & h(7h+3) & 3h \end{pmatrix}.$$

Then

$$\det(\widehat{\boldsymbol{\lambda}_h^T \mathbf{q}}) = (h-1)(2h+1)(2h-1)(1+h)^2,$$

hence

$$\det(\widehat{\boldsymbol{\lambda}_h^T \mathbf{q}}) \neq 0, \quad 0 < |h| < \frac{1}{2}.$$

Consequently,  $P$  is the pointwise limit of Lagrange projector  $P_h$ ,  $0 < |h| < \frac{1}{2}$ , as  $h$  tends to zero, with the property that  $\text{ran}P'_h = \text{span}_{\mathbb{F}}\boldsymbol{\lambda}_h$ .

More precisely, we select a test function

$$f(x_1, x_2, x_3) = 1 + (1 - x_1)^2 + (1 - x_2)^2 + (1 - x_3)^2$$

to describe the perturbation procedure for the projector in this example.

When  $h = 1/10, 1/100, 1/1000, \dots$ , we have

$$\begin{aligned} P_{\frac{1}{10}}f &= 4 - \frac{34949}{14520}x_3 - \frac{439}{7260}x_2 - \frac{37867}{9680}x_1 - \frac{2303}{2904}x_3^2 + \frac{233}{1452}x_3x_2 + \frac{7767}{1936}x_3x_1, \\ P_{\frac{1}{100}}f &= 4 - \frac{2600449499}{1274614950}x_3 - \frac{46747801}{2549229900}x_2 - \frac{483294631}{121391900}x_1 - \frac{24977753}{25492299}x_3^2 \\ &\quad + \frac{722401}{25492299}x_3x_2 + \frac{9690171}{2427838}x_3x_1, \\ P_{\frac{1}{1000}}f &= 4 - \frac{251000494994999}{125249623999500}x_3 - \frac{496749753001}{250499247999000}x_2 - \frac{333833081249167}{83499749333000}x_1 \\ &\quad - \frac{249997752503}{250499247999}x_3^2 + \frac{747249001}{250499247999}x_3x_2 + \frac{667833161997}{166999498666}x_3x_1, \\ &\dots \\ Pf &= 4 - 2x_3 - 4x_1 - x_3^2 + 4x_3x_1. \end{aligned}$$

□

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